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The Potts model and flows: I. The pair correlation function

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Abstract. It is shown that the partition function for the λ-state Potts model with pair interactions is related to the expected number of integer mod-λ flows in a percolation model. The relation is generalised to the pair correlation function. The resulting high-temperature expansion coefficients are shown to be the flow polynomials of graph theory. We also prove an observation of Tsallis and Levy concerning the equivalent transmissivity of a cluster.

1. Introduction

The Potts model (Potts 1952) has recently been reviewed by Wu (1982). Here we show that the thermal equilibrium properties of a Potts model with λ states per spin may be determined by counting mod-λ flows (see § 2 for definition). It turns out (§ 3) that the partition function is proportional to the expected number of such flows on the clusters of a percolation model in which the probability of an open (present) bond is related to the Potts interaction parameter. Similarly the spin pair correlation function is related to the expected number of flows conditional upon there being a path between the spin sites in the corresponding percolation problem.

Tsallis and Levy (1981) have defined an 'equivalent transmissivity' between spin pairs which we show to be related to the pair correlation function above and hence to the expected number of flows. This relation enables us to prove a number of properties of the equivalent transmissivity.

Domb (1974) has studied the derivation of high temperature expansions for the partition function. He found that the coefficients could be written as a weighted sum over star subgraphs of the lattice and that the weights were topological invariants. In § 4 we extend these results to the pair correlation function and show that these invariants are determined by counting the number of proper flows (see § 2) which can exist on the subgraph. This combinatorial problem appeared in the graph theory literature some time ago (Tutte (1954); in this paper proper flows were known as colour cycles). The number of proper flows is a polynomial in λ known as the 'flow polynomial' (Rota 1966, Tutte 1984).

Flow polynomials also arose in the field theoretic treatment of the Potts model by Amit (1976), but again only examples were given and the connection with flow polynomials (see § 5) seems not to have been recognised.

2. Flows and the flow polynomial

Let G be a graph with vertex set $V=(v_1, v_2, \ldots, v_{\nu(G)})$ and edge set $E=(e_1, e_2, \ldots, e_{\varepsilon(G)})$ (for a recent introduction to graph theory see Bollobas (1979)). Let $\Phi=(\Phi_1, \Phi_2, \ldots, \Phi_{\varepsilon(G)})$ be a vector, the jth component of which is associated with the edge e_j, and consider an arbitrary directing of the edges of E as indicated by the matrix

$$D_{ij}=\begin{cases} 1 & \text{if } e_j \text{ is directed away from } v_i, \\ -1 & \text{if } e_j \text{ is directed towards } v_i, \\ 0 & \text{if } e_j \text{ is not incident with } v_i. \end{cases} \tag{2.1}$$

The edge directing corresponds to the arbitrary choice of direction for cartesian axes. The values of the components of Φ are restricted by the flow condition

$$\sum_{j=1}^{\varepsilon(G)} D_{ij}\Phi_j=0 \qquad \text{for each } i \tag{2.2}$$

and vectors satisfying this condition (i.e. the signed sum of the Φ_j at each vertex is zero) are called flows. For example, the currents in an electrical network are flows and in this context are flow condition corresponds to charge conservation. Electrical currents can take on any real value, but in the context of the Potts model the Φ_j are only allowed the integer values $0, 1, 2, \ldots, \lambda-1$ and the arithmetic implied by the flow condition is done modulo λ. In this case, for given λ, the number of flows is finite and may be counted as follows.

A cycle on G is defined to be a subgraph of G which is either polygonal or a loop (an edge connecting a vertex to itself). The length of a cycle is the number of vertices it contains. A pair of parallel edges in a graph with repeated edges (multigraph) is a cycle of length two and in a simple graph all cycles have length at least three. We suppose that each cycle of G is given an arbitrary orientation. A flow of strength f in a cycle c is a flow such that if $e_j \in c$ then $\Phi_j=f$ or $\lambda-f$ depending on whether or not e_j is directed parallel or antiparallel to c, else $\Phi_j=0$. The strength of such a flow is determined by its value on some chosen edge of the cycle and can therefore have one of the λ values $0, 1, \ldots, \lambda-1$. A flow of unit strength in some cycle is called a primitive flow.

The set of all possible mod-λ flows on some graph G forms a vector space (the cycle space of G) and independent flows are independent vectors in this space. A spanning forest is a subgraph of G each tree of which spans all the vertices of one of the $\omega(G)$ components of G. The primitive flows in the cycles obtained by adding each of the possible additional edges to a spanning forest form an independent set. This is because each cycle has an edge which is not present in any of the other cycles. It may also be shown that these primitive flows form a basis in the cycle space (Biggs 1974). The number of flows, $c(G)$, in this basis is equal to the number of edges in the complement of a spanning forest and hence

$$c(G)=\varepsilon(G)-\nu(G)+\omega(G). \tag{2.3}$$

$c(G)$ is known as the cyclomatic number of G or the number of independent cycles in G. All possible flows may be generated by taking a linear combination of these $c(G)$ primitive flows, which is equivalent to assigning a strength to the flow in each one of the $c(G)$ cycles generated by some spanning forest. The strength of the flow in any cycle may be taken as the value of Φ on the edge of the cycle which does not belong to the spanning forest. We conclude that the total number of mod-λ flows is therefore $\lambda^{c(G)}$. In table 1 we give the nine mod-3 flows which can occur on the graph of figure 1 which has two independent cycles.

In the subsequent sections we shall find that it is the number of proper mod-λ flows, $F(\lambda, G)$, which determines the high-temperature expansion coefficients of the Potts model. A proper flow is one which has non-zero value on every edge. In the example above only flows six and eight are proper. Counting such flows on a general graph is a non-trivial combinatorial problem. A formula which gives the number of proper flows in terms of the number of flows on each subgraph of G may be obtained by an inclusion and exclusion argument (Rota 1966). Let A_j be the subset of flows which are zero on the edge e_j. $F(\lambda, G)$ is the number of flows which are zero on no edge and by inclusion and exclusion

$$F(\lambda, G) = \sum_{E''\subseteq E} (-1)^{|E''|} \left| \bigcap_{e_j \in E} A_j \right|. \tag{2.4}$$

But $|\ldots|$ is the number of flows which are zero on the edges of E'' and is therefore equal to the total number of flows on the subgraph G' of G which has edge set $E' = E\backslash E''$ and hence

$$F(\lambda, G) = \sum_{E'\subseteq E} (-1)^{|E\backslash E'|} \lambda^{c(G')}. \tag{2.5}$$

We see that $F(\lambda, G)$ is a polynomial in λ and is known as the flow polynomial of G. The graph of figure 1 has flow polynomial $(\lambda-1)(\lambda-2)$ which has the value zero for

Table 1. Mod-3 flows on the graph of figure 1. f is the strength of the flow in cycle c_i.

	f_1	f_2	$\Phi_1 = \Phi_2$	$\Phi_3 = \Phi_4$	Φ_5
1	0	0	0	0	0
2	0	1	0	1	2
3	0	2	0	2	1
4	1	0	1	0	1
5	1	1	1	1	0
6	1	2	1	2	2
7	2	0	2	0	2
8	2	1	2	1	1
9	2	2	2	2	0

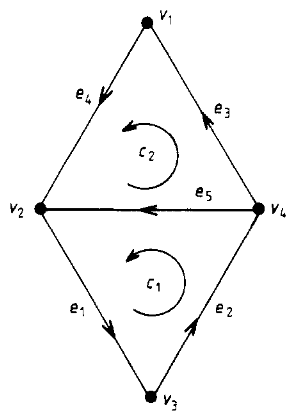


Figure 1. A graph with two independent cycles, graph 2.1 of table 2.

$\lambda = 1$ and 2, coresponding to the impossibility of proper mod-1 and mod-2 flows, and the value two for $\lambda = 3$ corresponding to the two proper mod-3 flows found above. In fact no graph has a proper mod-1 flow since the only allowed value of Φ_i is zero. Also only Euler graphs (graphs with all vertices of even degree) can have a proper mod-2 flow since the value of such a flow must be one on every edge.

It is clear that the number of flows is a topological invariant (i.e. is unchanged when any edge is replaced by two or more edges in series) and we list the flow polynomials for all distinct topological types with $\leqslant 5$ cycles in table 2. The polynomials are expressed as linear combinations of the factorial functions $(\lambda - 1)_i = (\lambda - 1)(\lambda - 2) \ldots (\lambda - i)$, $i = 1, \ldots, c(G)$ and the corresponding coefficients a_i are tabulated. Drawings of the graphs may be found in Heap (1966). The graphs have an

Table 2. Flow polynomials for graphs with $\leqslant 5$ independent cycles.

$$F(\lambda, G) = \sum_{i=1}^{c} a_i (\lambda - 1)_i.$$

G	a_1	a_2	a_3	a_4	a_5	$k(G)$	G	a_1	a_2	a_3	a_4	a_5	$k(G)$
1.1	1					1	5.49	0	3	11	7	1	1
2.1	0	1				−1	5.50	0	3	11	7	1	1
3.1	1	2	1			1	5.51	0	0	7	6	1	2
3.2	0	1	1			1	5.52	0	1	7	6	1	1
3.3	0	1	1			1	5.53	0	1	7	6	1	1
3.4	0	0	1			2	5.54	0	1	7	6	1	1
4.1	0	5	5	1		−1	5.55	0	1	7	6	1	1
4.2	0	3	4	1		−1	5.56	0	0	4	5	1-	2
4.3	1	4	4	1		−1	5.57	0	3	11	7	1	1
4.4	0	1	3	1		−1	5.58	0	1	7	6	1	1
4.5	0	3	4	1		−1	5.59	0	1	7	6	1	1
4.6	0	1	3	1		−1	5.60	0	1	7	6	1	1
4.7	0	1	3	1		−1	5.61	0	2	8	6	1	2
4.8	0	2	3	1		−2	5.62	0	2	8	6	1	2
4.9	0	1	3	1		−1	5.63	0	0	4	5	1	2
4.10	0	1	3	1		−1	5.64	0	1	7	6	1	1
4.11	0	2	3	1		−2	5.65	0	0	4	5	1	2
4.12	0	1	2	1		−3	5.66	0	0	4	5	1	2
4.13	0	1	3	1		−1	5.67	0	0	4	5	1	2
4.14	0	1	3	1		−1	5.68	0	1	7	6	1	1
4.15	0	0	2	1		−2	5.69	0	0	5	5	1	4
4.16	0	0	1	1		−4	5.70	0	2	6	5	1	4
4.17	0	1	1	1		−5	5.71	0	1	6	5	1	5
5.1	1	10	20	9	1	1	5.72	0	1	7	6	1	1
5.2	0	5	15	8	1	1	5.73	0	0	4	5	1	2
5.3	1	8	16	8	1	1	5.74	0	1	5	5	1	3
5.4	0	7	16	8	1	1	5.75	0	1	5	5	1	3
5.5	0	3	11	7	1	1	5.76	0	2	8	6	1	2
5.6	0	3	11	7	1	1	5.77	0	0	6	5	1	6
5.7	0	5	15	8	1	1	5.78	0	1	6	5	1	5
5.8	0	3	11	7	1	1	5.79	0	2	6	5	1	4
5.9	0	2	11	7	1	2	5.80	0	1	7	6	1	1
5.10	0	3	11	7	1	1	5.81	0	1	7	6	1	1
5.11	0	3	11	7	1	1	5.82	0	0	4	5	1	2
5.12	0	3	11	7	1	1	5.83	0	1	7	6	1	1
5.13	0	5	12	7	1	1	5.84	0	1	7	6	1	1

Table 2. (continued)

G	a_1	a_2	a_3	a_4	a_5	$k(G)$	G	a_1	a_2	a_3	a_4	a_5	$k(G)$
5.14	0	4	12	7	1	2	5.85	0	1	7	6	1	1
5.15	1	8	16	8	1	1	5.86	0	0	4	5	1	2
5.16	1	8	13	7	1	1	5.87	0	0	4	5	1	2
5.17	1	7	13	7	1	2	5.88	0	1	7	6	1	1
5.18	0	1	7	6	1	1	5.89	0	0	4	5	1	2
5.19	0	1	7	6	1	1	5.90	0	0	4	5	1	2
5.20	0	3	11	7	1	1	5.91	0	0	4	5	1	2
5.21	0	3	11	7	1	1	5.92	0	0	2	4	1	4
5.22	0	0	7	6	1	2	5.93	0	0	2	4	1	4
5.23	0	1	7	6	1	1	5.94	0	1	3	4	1	5
5.24	0	3	11	7	1	1	5.95	0	1	7	6	1	1
5.25	0	1	7	6	1	1	5.96	0	0	4	5	1	2
5.26	0	1	7	6	1	1	5.97	0	1	5	5	1	3
5.27	0	2	8	6	1	2	5.98	0	1	5	5	1	3
5.28	0	1	7	6	1	1	5.99	0	0	3	4	1	6
5.29	0	1	8	6	1	3	5.100	0	0	2	4	1	4
5.30	0	2	8	6	1	2	5.101	0	1	4	4	1	7
5.31	0	3	11	7	1	1	5.102	0	0	4	4	1	8
5.32	0	3	11	7	1	1	5.103	0	1	7	6	1	1
5.33	0	1	7	6	1	1	5.104	0	1	7	6	1	1
5.34	0	5	12	7	1	1	5.105	0	1	7	6	1	1
5.35	0	1	7	6	1	1	5.106	0	0	4	5	1	2
5.36	0	4	9	6	1	2	5.107	0	0	4	5	1	2
5.37	0	2	8	6	1	2	5.108	0	1	7	6	1	1
5.38	0	2	8	6	1	2	5.109	0	0	4	5	1	2
5.39	0	1	7	6	1	1	5.110	0	0	4	5	1	2
5.40	0	3	9	6	1	3	5.111	0	0	2	4	1	4
5.41	0	2	9	6	1	4	5.112	0	0	2	4	1	4
5.42	0	1	7	6	1	1	5.113	0	1	3	4	1	5
5.43	0	1	7	6	1	1	5.114	0	0	2	4	1	4
5.44	0	0	4	5	1	2	5.115	0	0	1	3	1	8
5.45	0	1	7	6	1	1	5.116	0	0	2	3	1	10
5.46	0	0	4	5	1	2	5.117	0	1	3	3	1	11
5.47	0	1	5	5	1	3	5.118	0	0	3	3	1	12
5.48	0	0	5	5	1	4							

identifier of the form $c \cdot x$ where c is the number of cycles and x is Heap's label. Graph 2.1 is the graph in figure 1. Notice that the graphs for which $a_1 = 0$ have polynomials with a factor $(\lambda-1)(\lambda-2)$ and have at least one vertex of odd degree since otherwise a proper mod-2 flow would be possible. Notice also that the last coefficient for every graph has value one; this results from the fact that the coefficient of $\lambda^{c(G)}$ is unity corresponding to the term $E' = E$ in (2.5). Graph 4.12, which is a square pyramid, has $F(\lambda, G) = (\lambda-1)_2 + 2(\lambda-1)_3 + (\lambda-1)_4$. The column headed $k(G)$ is the k-weight of the graph and is the weight which occurs in the expansion of the mean number of percolation clusters. (See Essam and Sykes (1966) who gave the values of k for $c \leqslant 4$. The values for $c = 5$ were given by Heap (1966).) We shall see (§ 4) that $k(G)$ is the value of $(\mathrm{d}/\mathrm{d}\lambda)F(\lambda, G)$ at $\lambda = 1$ and therefore serves as a check on the coefficients a_i in the table.

3. Thermal properties, colourings and flows

The partition function of the λ-state Potts model may be defined by

$$Z = \mathrm{Tr}\left[\exp\left(\sum_{[i,j]\in B} K_{ij} s_i \cdot s_j\right)\right] \tag{3.1}$$

where B is the set of interacting spin pairs or 'bonds' and the trace is over all positions of the ν spin vectors s_i. Each vector takes on one of the λ values which are the position vectors of the corners of a $(\lambda-1)(=n)$-dimensional tetrahedron relative to its centre (Amit 1976). If e_α and e_β are two such vectors then

$$e_\alpha \cdot e_\beta = (s^2/n)[\lambda\delta(\alpha,\beta)-1] \tag{3.2}$$

where $s=|s_i|$, and we shall suppose that $s^2=n$. Using this relation leads to

$$Z = \exp\left(n \sum_{[i,j]\in B} K_{ij}\right)\Lambda \tag{3.3}$$

where the reduced partition function Λ is given by

$$\Lambda = \sum_{\alpha} \prod_{[i,j]\in B} z_{ij}^{1-\delta(\alpha_i,\alpha_j)} \tag{3.4}$$

where $\alpha_i = 1, \ldots, \lambda$ and $z_{ij} = \exp(-\lambda K_{ij})$. Carrying out the α sums gives the well known result (Kasteleyn and Fortuin 1969)

$$\Lambda = \sum_{B'\subseteq B} \lambda^{\omega(B')} \prod_{b\in B'} p_b \prod_{b\in B\setminus B'} (1-p_b) \tag{3.5}$$

where $\omega(B')$ is the number of clusters into which the spin sites are partitioned by the subset B' of bonds and $p_b = 1-z_{ij}$. As K_{ij} varies from 0 to ∞, p_b goes from 0 to 1 and can be interpreted as the probability of an open bond in a bond percolation process (Kasteleyn and Fortuin 1969). Thus Λ is the expected number of ways that the spin sites may be coloured in one of λ colours such that all the sites in the same percolation cluster are coloured alike.

A second probabilistic interpretation of Λ is in terms of mod-λ flows. To make contact with § 2, here G is the graph with V equal to the set of spins and $E = B$, the set of interactions. After some manipulation, equation (3.5) becomes

$$\Lambda = \lambda^{\nu(G)} \prod_{b\in B} (1+nt_b)^{-1} E(\lambda^c) \tag{3.6}$$

where

$$t_b = \frac{p_b}{\lambda - np_b} = \frac{1-z_b}{1+nz_b} \tag{3.7}$$

and the non-trivial factor $E(\lambda^c)$ is given by

$$E(\lambda^c) = \sum_{B'\subseteq B} \lambda^{c(G')} \prod_{b\in B'} t_b \prod_{b\in B\setminus B'} (1-t_b) \tag{3.8}$$

where $c(G')$ is the number of independent cycles in the subgraph G' defined by the bonds B'. We note that $\lambda^{c(G')}$ is the number of possible mod-λ flows on the subgraph G'. Thus the right-hand side of (3.8) may be interpreted, for $K_b \geq 0$, as the expected number of mod-λ flows which may be found on the clusters of a bond percolation process in which the bond b occurs with probability t_b (hence the notation). The probabilistic nature of the variable t_b has already been emphasised by Tsallis and Levy (1981).

A similar result may be obtained for the correlation function $\langle s_i \cdot s_j \rangle$ by introducing an extra bond g ('ghost bond' (Kasteleyn and Fortuin 1969)) between the spins i and j thus:

$$\langle s_i \cdot s_j \rangle = \frac{\partial}{\partial K_g} \log Z \bigg|_{K_g=0} \tag{3.9}$$

$$= \frac{\partial}{\partial K_g} \log E(\lambda^c) \bigg|_{K_g=0} \tag{3.10}$$

$$= \frac{\partial}{\partial t_g} \log E(\lambda^c) \bigg|_{t_g=0}. \tag{3.11}$$

The expectation value in (3.11) is defined by (3.8) with B replaced by $B^+ = B \cup g$. If the sum over $B' \subseteq B^+$ is split into two terms according to whether $g \in B'$ or $g \in B^+ \backslash B'$ then

$$\frac{\partial}{\partial t_g} E(\lambda^c) \bigg|_{t_g=0} = \sum_{B' \subseteq B} (\lambda^{c(G' \cup g)} - \lambda^{c(G')}) \prod_{b \in B'} t_b \prod_{b \in B \backslash B'} (1 - t_b) \tag{3.12}$$

and since

$$\lambda^{c(G' \cup g)} - \lambda^{c(G')} = \gamma_{ij}(G') \lambda^{c(G')} (\lambda - 1) \tag{3.13}$$

where $\gamma_{ij}(G') = 1$ or 0 depending on whether or not there is a path connecting i and j which uses only bonds of B', we have

$$\langle s_i \cdot s_j \rangle = n E(\gamma_{ij} \lambda^c) / E(\lambda^c). \tag{3.14}$$

The factor $E(\gamma_{ij}\lambda^c)$ is the expected number of integer mod-λ flows given that at least one path occurs connecting i and j, times the probability of such an occurrence.

In real space renormalisation group calculations (e.g. Yeomans and Stinchcombe (1980), Tsallis and Levy (1981); for a review see Tsallis (1985)) it is necessary to replace a cluster of spins by a single equivalent bond connecting just two of the spins s_1 and s_2 (say). The equivalent interaction K_{eq} is computed by tracing over all the other spins thus:

$$\text{Tr}' \left[\exp\left(\sum_{[i,j] \in B} K_{ij} s_i \cdot s_j \right) \right] = C \exp(K_{\text{eq}} s_1 \cdot s_2). \tag{3.15}$$

Carrying out the trace of (3.15) over s_1 and s_2 with and without the factor $s_1 \cdot s_2$ and taking the ratio of the results gives

$$\langle s_1 \cdot s_2 \rangle = n t_{12}^{\text{eq}} \tag{3.16}$$

with t_{12}^{eq} related to K_{eq} by (3.7). Combining (3.14) and (3.16) we obtain

$$t_{12}^{\text{eq}} = E(\gamma \lambda^c) / E(\lambda^c) \tag{3.17}$$

where $\gamma = \gamma_{12}$. It follows from (3.2) and (3.16) that t_{12}^{eq} is the probability that spin 2 is in the same state as spin 1 minus the probability that it is in some particular different state. It is what Tsallis and Levy (1981) call the equivalent transmissivity of spins 1 and 2 of the cluster. t_b is the transmissivity of the bond b and we note that, from their definitions, the numerator N and denominator D of (3.17) are multilinear functions of the bond transmissivities. Also $D = 1$ when all interactions are set to zero. We therefore conclude that N and D are precisely the functions defined by Tsallis and Levy which we have now interpreted as expected values in a bond percolation model.

The equivalent transmissivity of any spin pair i,j is $\langle s_i \cdot s_j \rangle / n$ and has numerator and denominator given by

$$N_{ij} = E(\gamma_{ij}\lambda^c), \qquad D = E(\lambda^c). \tag{3.18}$$

Notice that D is independent of i and j and also determines the partition function. Setting $t_b = 1$ for all b in (3.8) makes all the terms in the sum equal to zero except the term $B' = B$ which proves the observation of Tsallis and Levy (1981) that the sum of their denominator coefficients is equal to $\lambda^{c(G)}$. The same is true of the numerator coefficients provided that spins i and j are connected by an interaction path.

4. High temperature expansion and related properties of the partition and correlation functions

We next turn to the derivation of power series expansions for the Potts model. Domb (1974) has given a detailed discussion of the graph theoretic properties of the coefficients which occur when Z is expanded in the variable t_b (equation (3.7)). Here we relate these coefficients to the number of proper mod-λ flows (§ 2) on the clusters of B'. The flow constraint results in many of the subsets B' having zero weight, a fact which is not immediately obvious from previous formulations.

Expanding the last product in (3.8) gives

$$E(\lambda^c) = \sum_{B' \subseteq B} \lambda^{c(G')} \prod_{b \in B'} t_b \sum_{B'' \subseteq B \setminus B'} (-1)^{|B''|} \prod_{b \in B''} t_b \tag{4.1}$$

$$= \sum_{B' \subseteq B} F(\lambda, G') \prod_{b \in B'} t_b \tag{4.2}$$

where we have replaced $B' \cup B''$ by B', inverted the order of summation, and used (2.5). When the spins lie on a crystal lattice, if B is the set of nearest-neighbour bonds then $t_b = t$ for all b and (4.2) is an expansion in powers of t. For an infinite lattice, (4.2) is an infinite series which converges for sufficiently high temperatures. Many of the properties of the coefficients in this expansion have been discussed by Domb (1974) but the interpretation in terms of flow polynomials was not given. Equation (4.2) may be deduced from the work of Biggs (1976, 1977) who expressed the Potts partition function as a sum over flows. Differentiating (4.1) with respect to λ and setting $\lambda = 1$ gives the expansion of $E(c)$, the mean number of independent cycles in the clusters of a bond percolation model. This is simply related to $E(\omega)$, the mean number of clusters, by (2.3) which is valid for any subgraph. The relation between the coefficients in the mean number expansion and the derivative of F, mentioned in § 2, follows immediately.

Extension of (4.2) to the expansion of $E(\gamma_{ij}\lambda^c)$ follows in a similar manner by first introducing the factor $\gamma_{ij}(B')$ into (3.8):

$$E(\gamma_{ij}\lambda^c) = \sum_{B' \subseteq B} F_{ij}(\lambda, G') \prod_{b \in B'} t_b \tag{4.3}$$

where

$$F_{ij}(\lambda, G') = \sum_{B'' \subseteq B'} (-1)^{|B' \setminus B''|} \gamma_{ij}(B'') \lambda^{c(G'')}. \tag{4.4}$$

In the calculation of the correlation between spins s_i and s_j of the cluster the vertices i and j are special and are called roots. We therefore call $F_{ij}(\lambda, G')$ the rooted flow polynomial of G' relative to the roots i and j. $F_{ij}(\lambda, G')$ may be related to $F(\lambda, G' \cup g)$, where g is the 'ghost bond' introduced in § 3, as follows. From (2.5)

$$F(\lambda, G' \cup g) = \sum_{B'' \subseteq B' \cup g} (-1)^{|B' \cup g \backslash B''|} \lambda^{c(G'')}. \tag{4.5}$$

Dividing this sum into four parts depending on whether or not $\gamma_{ij}(B'') = 1$ and whether or not $g \in B''$ shows that the terms for which $\gamma_{ij}(B'') = 0$ cancel, and combining the other two terms gives

$$F_{ij}(\lambda, G') = n^{-1} F(\lambda, G' \cup g) \tag{4.6}$$

which is the number of proper flows in $G' \cup g$ with a fixed non-zero value on g. As an example, suppose we require $F_{13}(\lambda)$ for the graph of figure 1. Adding the ghost bond gives the complete graph on four vertices (tetrahedron) which is graph 3.4 of table 2 and has flow polynomial $(\lambda-1)_3$. Using (4.6) gives $F_{13}(\lambda) = (\lambda-2)(\lambda-3)$. Thus there are two rooted proper mod-4 flows; if there is an external flow in at v_1 and out at v_2 of strength one, then the two flows in question are (3, 2, 1, 2, 1) and (2, 1, 2, 3, 3). Equations (4.2) and (4.3) are illustrated in figure 2.

The following properties of $F(\lambda, G)$ follow in an intuitively obvious manner from the conservation of 'fluid' (mod λ) at each vertex of G and the condition that there must be a non-zero flow in each edge. A formal proof of most of these properties may be found in Tutte (1954, 1984).

(i) $F = 0$ if G has a vertex of degree one or an articulation edge (isthmus).

(ii) If G has components $G_1, \ldots, G_m$ then

$$F(\lambda, G) = \prod_{r=1}^{m} F(\lambda, G_r). \tag{4.7}$$

The same result is true even if the G_i have articulation vertices in common.

(iii) F is a topological invariant, that is graphs which are homeomorphic (isomorphic when vertices of degree two are suppressed) have the same value of F.

(iv) Deletion-contraction rule. If the edge e of G is not a loop

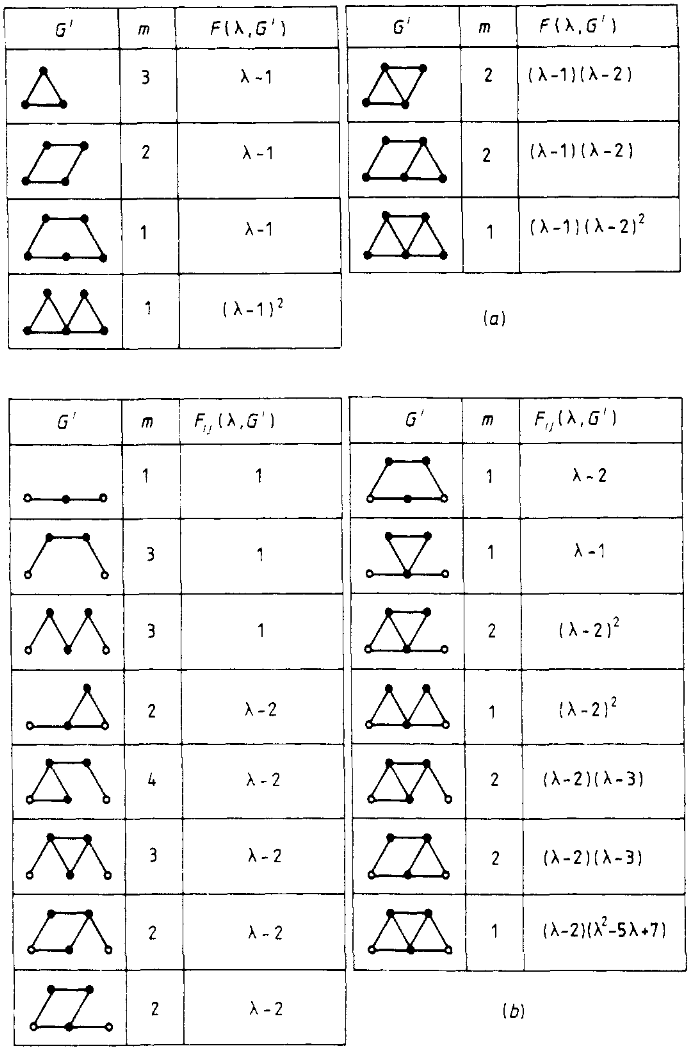
$$F(\lambda, G) = F(\lambda, G_e^{\gamma}) - F(\lambda, G_e^{\delta}) \tag{4.8}$$

where G_e^{δ} and G_e^{γ} are obtained by deleting and contracting respectively the edge e. This result is not so obvious but follows by noting that the flows on G_e^{δ} correspond to flows on G which are zero on e but non-zero on all other edges. The flows on G_e^{γ} correspond to flows on G which are non-zero everywhere except possibly on e. Although loops have no physical significance in Potts model applications they can arise if the deletion-contraction rule is repeatedly applied. If e is a loop then

$$F(\lambda, G) = (\lambda-1) F(\lambda, G_e^{\delta}). \tag{4.8a}$$

(v) Edge doubling. Most graphs with c cycles can be obtained by replacing an edge of a graph with $c-1$ cycles by a double edge. For example, 104 of the 118 graphs in table 2 with five cycles may be obtained in this way. Let G_{ef} be the graph obtained from G by replacing the edge e by the double edge e, f. The proper flows on G_{ef} are of two types depending on whether or not there is a net non-zero flow through the double edge. There are $\lambda-2$ flows of the first type for every proper flow on G and $\lambda-1$ of the second type for each proper flow on G_e^{δ}, thus

$$F(\lambda, G_{ef}) = (\lambda-2) F(\lambda, G) + (\lambda-1) F(G_e^{\delta}). \tag{4.9}$$



G′	m	F(λ, G′)
	3	λ-1
	2	λ-1
	1	λ-1
	1	$(\lambda-1)^2$

G′	m	F(λ, G′)
	2	(λ-1)(λ-2)
	2	(λ-1)(λ-2)
	1	$(\lambda-1)(\lambda-2)^2$

(a)

G′	m	$F_{ij}(\lambda, G')$
	1	1
	3	1
	3	1
	2	λ-2
	4	λ-2
	3	λ-2
	2	λ-2
	2	λ-2

G′	m	$F_{ij}(\lambda, G')$
	1	λ-2
	1	λ-1
	2	$(\lambda-2)^2$
	1	$(\lambda-2)^2$
	2	(λ-2)(λ-3)
	2	(λ-2)(λ-3)
	1	$(\lambda-2)(\lambda^2-5\lambda+7)$

(b)

Figure 2. Calculation of the pair correlation function for a cluster using flow polynomials. The interactions represented by the edges are assumed equal and only non-isomorphic subgraphs are shown. m is the number of subgraphs which are isomorphic to the graph shown. (a) yields the denominator $D(\;) = 1+(\lambda-1)(3t^3+2t^4+t^5)+(\lambda-1)^2t^6+(\lambda-1)(\lambda-2)(2t^5+2t^6)+(\lambda-1)(\lambda-2)^2t^7$; ($b$) yields the numerator $N_{ij}(\;) = t^2+3t^3+3t^4+(\lambda-2)(2t^4+12t^5)+(\lambda-1)t^5+[3(\lambda-2)^2+4(\lambda-2)(\lambda-3)]t^6+(\lambda-2)(\lambda^2-5\lambda+7)t^7$.

The prefactors are the number of proper flows in the extra cycle formed by the double edge.

Similar properties follow for F_{ij} using the connection (4.6) with F and are listed below with the corresponding numbering.

(i) $F_{ij}=0$ if G has a vertex of degree one which is not i or j.

(ii) If G has components $G_1, \ldots, G_m$ and if the roots i and j both belong to G_1 then

$$F_{ij}(\lambda, G) = F_{ij}(\lambda, G_1) \prod_{r=2}^{m} F(\lambda, G_r). \tag{4.7a}$$

The same result is true even if the G_i have articulation vertices in common.

If i and j are in different components then $F_{ij} = 0$.

(iii) Two rooted flow polynomials are the same whenever the corresponding rooted graphs are homeomorphic (rooted graphs are isomorphic if there is a bijection between their vertex sets which induces a correspondence between the rooted vertices and the edges of the two graphs).

(iv) Equations (4.8) and (4.8*a*) are valid with F replaced by F_{ij}.

(v) Equation (4.9) is valid with F replaced by F_{ij}.

A particularly interesting relation for F_{ij}, the 'onion property', may be deduced from (4.8). Consider the vertex i of a graph G and suppose that the edges incident with i are partitioned into two non-empty subsets E_1 and E_2. Now let G be 'cut through' vertex i so as to separate these subsets (figure 3), leaving i incident with E_1 and introducing a new vertex j which is incident with E_2. This operation gives rise to a family of new graphs G_k, $k = 1, 2, \ldots$, each one corresponding to a different partition. These new graphs can be transformed into one another by an operation which is like peeling an onion.

(vi) The 'onion' property implies

$$F(\lambda, G_k) + (\lambda - 1)F_{ij}(\lambda, G_k) = F(\lambda, G) \tag{4.10}$$

and hence that the left-hand side is independent of k.

The result is illustrated in figure 3. To prove this result we use (4.6) to replace $(\lambda - 1)F_{ij}(\lambda, G_k)$ by $F(\lambda, G_k \cup g)$ and then apply the deletion-contraction rule to the edge g of $G_k \cup g$. Equation (4.10) is also trivially true for the graph G_0, corresponding to the case E_2 is empty, since then j is just an additional isolated vertex and $F_{ij} = 0$.

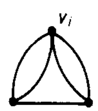
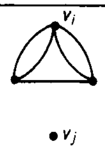
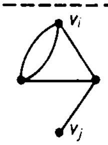
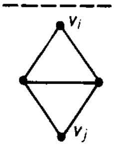
Graph	$F(\lambda, G)$	$F_{ij}(\lambda, G)$
G 	$(\lambda-1)(\lambda-2)^2$	—
G_0 	$(\lambda-1)(\lambda-2)^2$	0
G_1 	0	$(\lambda-2)^2$
G_2 	$(\lambda-1)(\lambda-2)$	$(\lambda-2)(\lambda-3)$

Figure 3. The family of graphs obtained by 'peeling' G at v_i (only non-isomorphic graphs are shown).

The following properties of N_{ij} and D may be deduced from the corresponding properties of F and F_{ij} using (3.18), (4.2) and (4.3).

(i) Vertices of degree one may be deleted from G without changing $D(G)$. The same is true of $N_{ij}(G)$, and hence $\langle s_i \cdot s_j \rangle$, with the exception of vertices i and j.

(ii) With G_i as in (ii) above

$$D(G) = \prod_{r=1}^{m} D(G_r) \tag{4.11}$$

and if i and j are both in G_1

$$N_{ij}(G) = N_{ij}(G_1) \prod_{r=2}^{m} D(G_r) \tag{4.12}$$

and

$$\langle s_i \cdot s_j \rangle = N_{ij}(G_1)/D(G_1). \tag{4.13}$$

This result also applies if the G_i have articulation vertices in common. If i and j are in different components then $N_{ij}(G) = 0$ since there can be no flows which are proper on the 'ghost' edge g.

(iii) If bonds b_1 and b_2, with transmissivities t_1 and t_2, are in series (i.e. have a non-rooted vertex of degree two in common) then, in calculating N_{ij} and D, they may be replaced by a single bond with transmissivity $t_1 t_2$.

(iv) The 'break-collapse method' of Tsallis and Levy (1981) is obtained by splitting the sum in (4.2) into two parts depending on whether or not the bond e is in B' and then applying the deletion-contraction rule to the flow polynomials in the first part, thus:

$$D(G) = \sum_{B' \subseteq B \setminus e} [F(\lambda, G') + t_e F(\lambda, G' \cup e)] \prod_{b \in B'} t_b \tag{4.14}$$

and hence

$$D(G) = t_e D(G_e^{\gamma}) + (1 - t_e) D(G_e^{\delta}). \tag{4.15}$$

The same argument, starting from (4.3), is valid for $N_{ij}(G)$. The terms on the right-hand side of (4.15) can be interpreted in terms of the expected numbers of flows given that the bond e is or is not present in the equivalent percolation problem. If e is a loop then

$$D(G) = [1 + (\lambda - 1) t_e] D(G_e^{\delta}). \tag{4.15a}$$

(v) If G_{ef} is the graph obtained by doubling the edge e of G then

$$D(G_{ef}) = [t_e + t_f + (\lambda - 2) t_e t_f][D(G_e^{\gamma}) - D(G_e^{\delta})] + [1 + (\lambda - 1) t_e t_f] D(G_e^{\delta}) \tag{4.16}$$

with a similar relation for N_{ij}. Again the flows which determine $D(G_{ef})$ have been subdivided according to whether or not there is a net flow between the double edge and the rest of the graph.

(vi) The 'onion property' follows by substituting (4.10) applied to G' into (4.2) and using (4.3) to give

$$D(G_k) + (\lambda - 1) N_{ij}(G_k) = D(G). \tag{4.17}$$

G'_k is obtained from G' using the partition induced by that which gave G_k from G.

5. Spin traces and flow polynomials

We now show that the flow polynomials may be used to evaluate spin traces. Since the scalar product of a pair of Potts model spin vectors takes on only two values, the exponential in (3.1) may be written

$$\exp(K_{ij}s_i \cdot s_j) = A_{ij}(1 + t_{ij}s_i \cdot s_j) \tag{5.1}$$

where t_{ij} is given by (3.7) and with $s^2 = n$

$$A_{ij} = \exp(nK_{ij})/(1 + nt_{ij}). \tag{5.2}$$

Substitution in (3.1) shows that

$$Z = \left(\prod_{b \in B} A_b\right) \mathrm{Tr}\left(\sum_{B' \subseteq B} \prod_{[i,j] \in B'} (t_{ij}s_i \cdot s_j)\right) \tag{5.3}$$

and using (3.3) and (3.6) gives (4.2) where now

$$F(\lambda, G') = \mathrm{Tr}\left(\prod_{[i,j] \in B'} (s_i \cdot s_j)\right) / \lambda^{\nu(G)}. \tag{5.4}$$

We note that G' has the same number of vertices $\nu(G)$ as G. It may therefore be concluded that this normalised spin trace is the flow polynomial of Tutte (1954). In view of the importance of such spin traces in Potts model theory (for example they also arise in the field theoretic formulation of renormalisation group theory (Amit 1976)), we present a direct derivation of (5.4)†.

Denote the right-hand side of (5.4) by $T(\lambda, G')$. Using (3.2) with $s^2 = n$ and replacing the trace by a sum over the state variables α_i gives

$$T(\lambda, G') = \lambda^{-\nu(G)} \sum_{\alpha} \prod_{[i,j] \in B'} [\lambda \delta(\alpha_i \cdot \alpha_j) - 1] \tag{5.5}$$

$$= \lambda^{-\nu(G)} \sum_{\alpha} \sum_{B'' \subseteq B'} (-1)^{|B' \setminus B''|} \prod_{[i,j] \in B''} [\lambda \delta(\alpha_i \cdot \alpha_j)] \tag{5.6}$$

$$= \sum_{B'' \subseteq B'} (-1)^{|B' \setminus B''|} \lambda^{\varepsilon(G'') + \omega(B'') - \nu(G'')}. \tag{5.7}$$

The factor $\lambda^{\omega(G'')}$ arises since the delta functions restrict the values of α_i to be equal for all spins in a given component. Using (2.3) applied to G'' gives

$$T(\lambda, G') = \sum_{B'' \subseteq B'} (-1)^{|B' \setminus B''|} \lambda^{c(G'')} \tag{5.8}$$

which confirms (5.4) using formula (2.5) for the number of proper flows on G'.

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